Solution to Problem Set 5

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Problem 1. Nash Equilibrium in a simple game

Two firms are deciding simultaneously whether to enter a market. If neither enters, they make zero profits. If both enter, they make profits -1, since the market is too small for two firms. If only one enters, that firm makes high profits. This game is summarized in the following matrix:

$1\backslash 2$	Enter	Do not Enter
Enter	$-1, -1$	$10, 0$
Do not Enter	$0, 5$	$0, 0$

Question

1. What are the pure-strategy Nash Equilibria of this game?
2. Now assume that firm 1 can enter the market with probability p_1 and firm 2 can enter the market with probability p_2. Write down the expected profits of each firm as a function of the strategy of the other player, and find the best response correspondence for firms 1 and 2.
3. Graph these best response correspondences and find the Nash equilibria in mixed strategies.
4. Is there one equilibrium out of these that seems more plausible to you.

Answer

1

Enumerate all possible actions:

- (Enter, Enter): Both firms have incentives to deviate to "Do not Enter"
- (Enter, Do not Enter): Both firms do not have incentives to deviate
- (Do not Enter, Enter): Both firms do not have incentives to deviate
- (Do not Enter, Do not Enter): Both firms have incentives to deviate to "Enter"

According to the definition of Nash Equilibrium, the two pure-strategy equilibria of this game is (Enter, Do not Enter) and (Do not Enter, Enter).

2

Because we assume that firm 1 can enter the market with probability p_1 and firm 2 can enter the market with probability p_2.

The expected profit of each firm 1 is:

$$EP_1 = \begin{cases} p_2 \cdot (-1) + (1 - p_2) \cdot (10) = 10 - 11p_2 & \text{if firm 1 chooses "Enter"} \\ p_2 \cdot (0) + (1 - p_2) \cdot (0) = 0 & \text{if firm 1 chooses "Do not Enter"} \end{cases}$$

Similarly, for firm 2, we have:

$$EP_2 = \begin{cases} p_1 \cdot (-1) + (1 - p_1) \cdot (5) = 5 - 6p_1 & \text{if firm 2 chooses "Enter"} \\ p_1 \cdot (0) + (1 - p_1) \cdot (0) = 0 & \text{if firm 2 chooses "Do not Enter"} \end{cases}$$

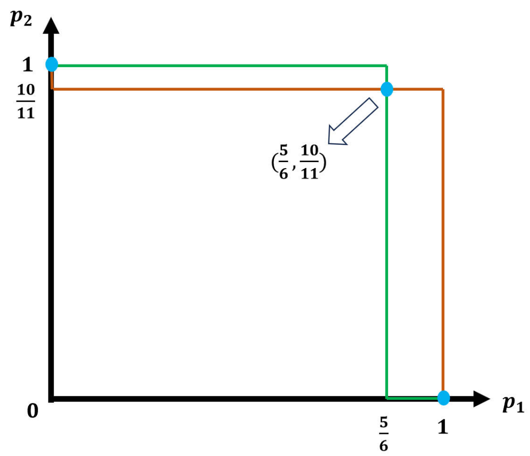
To find the best response correspondence (BR), we need to discuss according to the specific value of p_1 and p_2:

$$p_1^*(p_2) = \begin{cases} 1 & \text{if} \quad 0 \leqslant p_2 < \frac{10}{11} \\ \forall x \in [0, 1] & \text{if} \quad p_2 = \frac{10}{11} \\ 0 & \text{if} \quad \frac{10}{11} < p_2 \leqslant 1 \end{cases}$$

$$p_2^*(p_1) = \begin{cases} 1 & \text{if} \quad 0 \leqslant p_1 < \frac{5}{6} \\ \forall x \in [0, 1] & \text{if} \quad p_1 = \frac{5}{6} \\ 0 & \text{if} \quad \frac{5}{6} < p_1 \leqslant 1 \end{cases}$$

3

Graph these best response correspondences and we can easily find the Nash equilibria in mixed strategies located at the intersection.



From this graph, we can identify the two pure-strategy equilibria mentioned in part 1, and also a mixed-strategy equilibrium: firm 1 can enter the market with probability $p_1 = \frac{5}{6}$ and firm 2 can enter the market with probability $p_2 = \frac{10}{11}$, i.e. ($\frac{5}{6} \circ$ Enter $+ \frac{1}{6} \circ$ Do not Enter, $\frac{10}{11} \circ$ Enter $+ \frac{1}{11} \circ$ Do not Enter).

4

"Firm 1 enters while firm 2 remains outside the market", i.e. (Enter, Do not Enter), seems to be more plausible.

The market size is small so that at most 1 firm can survive in this market. However, the monopoly profits for firm 1 is 10, higher than the monopoly profits for firm 2. So, there are room for side payment from firm 1 to firm 2 so that firm 1 may prevent firm 2 from entering the market.

Nash equilibriium is a "non-cooperative" game in the sense that two firms cannot communicate with each other. In reality, if firms can communicate, it may makes the more profitable firm enter the market.

(This is an open question, so students can have their own opinions.)

Problem 2. Monopoly, Oligopoly and Perfect Competition

In this problem you are asked to compare the outcomes of monopoly, oligopoly, and perfect competition in one market. We are going to assume very simple functional forms in order to simplify the algebra. We assume that the firm has a very simple cost function: $c(y) = cy$ with $c > 0$. The marginal cost of production therefore is constant. As for the market demand, we assume that it takes the simple linear form $p(Y) = a - bY$, with $a > c > 0$ and $b > 0$, where Y is the total production on the industry.

Question

1. Consider first the case of perfect competition. Derive the marginal and average cost curves. How does the supply curve look like for each firm? What about in the industry? (aggregate the individual supply curve over J firms).

2. Equate supply and demand to obtain the industry-level production under perfect competition Y^*_{PC}, as well as the price level under perfect competition p^*_{PC}

3. How do perfect competition price and quantity vary as the cost of production c increases? How do they vary if there is a positive demand shock (a increases)?

4. We consider now the monopoly case. Write down the profit maximization problem and the first order conditions with respect to y. (In the case of monopoly, $y = Y$)

5. Solve for y^* and p^*_M. How does p^*_M vary as a increases? Why is this comparative statics different from the one under perfect competition?

6. Compare the total output and prices of perfect competition and monopoly. Compute the monopoly profits and compare them to the profits under perfect competition.

7. Consider now the case of duopoly, that is, an oligopoly with two firms, $i = 1, 2$. Write down the profit maximization problem of firm i as a function of the quantity produced by firm $-i$, y_{-i}.

8. We are now looking for Nash equilibria (in pure strategies) in the quantity produced y_1, y_2. Each firm i, holding fixed the quantity produced by the other firm at y^*_{-i}, maximizes profits with respect to y_i. Write down the first order conditions for firms 1 and 2.

9. In a Nash equilibrium each firm must choose the optimal quantity produced given the production choice of the other firm. Combine the two first order conditions in part 8 to obtain the Nash equilibrium quantity produced by firm 1, y^*_1, and by firm 2, y^*_2. Derive also the industry production, $Y^*_D = y^*_1 + y^*_2$, the price p^*_D, the profit level of each firm $\pi^*_{i,D}$, and the aggregate profit level $\Pi^*_D = \pi^*_{1,D} + \pi^*_{2,D}$.

10. Compare Y_D^*, Π_D^*, and p_D^* with Y_{PC}^*, Π_{PC}^*, and p_{PC}^* and Y_M^*, Π_M^*, and p_M^*.

11. Finally, the general case of oligopoly. Assume that there are I firms, all identical, with production costs as above. Write down the profit maximization problem and the first-order condition of a firm.

12. We now solve for the oligopoly production using a trick. We look for a symmetric solution, that is, a solution where each firm produces the same. In particular, impose the condition $y_{-i}^* = \sum_{j \neq i} y_j^* = (I-1)y_O^*$. Find the solution for the Nash equilibrium quantity y_O^*. Derive also the industry production $Y_O^* = Iy_O^*$, the price p_O^*, the profit level of each firm π_O^*, and the aggregate profit level $\Pi_O^* = I\pi_O^*$.

13. What is nice about this general oligopoly solution is that embeds the previous cases: for $I = 1$ we go back to monopoly, for $I = 2$ we get the duopoly solution. Most interestingly, show that for $I \to \infty$, the prices, the total quantity produced, and the total industry profits converge to the perfect competition ones. Compare this to Bertrand competition. How many companies did we need there to get the same outcomes as perfect competition?

Answer

1

The marginal cost curve and average cost curves are:

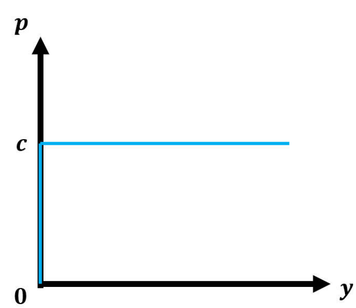
$$MC = \frac{\partial c(y)}{\partial y} = c \qquad AC = \frac{c(y)}{y} = c$$

In the context of perfect competition, the price p is exogenous given for any firm, hence, we should discuss the supply of each firm according to the value of p.

$$y = \begin{cases} 0 & \text{if} \quad p < c \\ \forall x \in [0, +\infty] & \text{if} \quad p = c \\ +\infty & \text{if} \quad p > c \end{cases}$$

When $p = c$, any positive output is optimal for the firm because they all result in the same profits.

The figure below shows the shape of the supply curve for any firm under perfect competition.



As for the whole industry, we need to aggregate the individual supply curve over J firms, which is:

$$Y = Jy = \begin{cases} 0 & \text{if} \quad p < c \\ \forall x \in [0, +\infty] & \text{if} \quad p = c \\ +\infty & \text{if} \quad p > c \end{cases}$$

2

The equilibrium price should set demand equal to supply:

- when $p < c$, there is always excess demand as no producers would like to produce and supply goods.

- when $p > c$, there is always excess supply because firms are able to produce infinite amount of goods.

Therefore, the only price that is possible in equilibrium is $p = c$, it is indeed balance demand and supply.

$$p^*_{PC} = c \Rightarrow p^*_{PC} = a - bY^*_{PC} \Rightarrow Y^*_{PC} = \frac{a-c}{b}$$

In conclusion, the industry-level production under perfect competition $Y^*_{PC} = \frac{a-c}{b}$, the price level under perfect competition $p^*_{PC} = c$.

3

According to $p^*_{PC} = c$ and $Y^*_{PC} = \frac{a-c}{b}$,

If c increases, the price will increase and the quantity will decrease.

If a increases (positive demand shock), the price will keep to be constant (c) and the quantity will increase.

4

The profit maximization problem in the monopoly case ($Y = y$) can be written as:

$$\max_{y} \quad \pi = p(y)y - cy = (a - by)y - cy = -by^2 + (a-c)y$$

The first order condition with respect to y is:

$$-2by + a - c = 0$$

5

From the FOC in part 4, we can get the optimal level of quantity and price:

$$y^* = \frac{a-c}{2b} \qquad p^*_M = a - by^* = \frac{a+c}{2}$$

As a increases, p^*_M will increase.

The difference in comparative statics between the monopoly and perfect competition cases:

In perfectly competitive market, competition will always drive price down to the marginal cost. In this constant marginal cost case, demand does not impact equilibrium price.

A monopoly can price based on demand elasticity. With the linear demand here, as a increases, for the same output, demand becomes more elastic. Therefore, the monopolistic firm will increase price.

6

Compare the total output and prices of perfect competition and monopoly as below:

	Perfect Competition	Monopoly
Total Output	$\frac{a-c}{b}$	$\frac{a-c}{2b}$
Price	c	$\frac{a+c}{2}$

The total output in the perfect competition case is higher than monopoly case, while the price in the perfect competition case is lower.

As for the the profits, under perfect competition $p = c$, the profit for any firm equals to zero, hence, the total profit is also zero ($\Pi^*_{PC} = 0$). For monopolistic firm, the profit is:

$$\Pi^*_M = \pi^*_M = p^*_M y^* - cy^* = (a - by^*)y^* - cy^* = \frac{(a-c)^2}{4b}$$

7

In the case of duopoly, there are two firms, the profit maximization problem of firm i as a function of the quantity produced by the firm $-i$, y_{-i} is:

$$\max_{y_i} \quad \pi_i = p(Y)y_i - cy_i = (a - bY)y_i - cy_i = [a - b(y_i + y_{-i})]y_i - cy_i$$

8

For Nash Equilibria (in pure strategies) in the quantity produced y_1 and y_2, we maximize the profits of each firm by choosing the optimal y_i while holding fixed the quantity produced by its opponent y^*_{-i}.

The first order conditions can be written as below:

$$-by_i + [a - b(y_i + y_{-i})] - c = 0$$

For firm 1:

$$-by_1 + [a - b(y_1 + y_2)] - c = 0 \Rightarrow y_1 = \frac{a-c}{2b} - \frac{1}{2}y_2$$

For firm 2:

$$-by_2 + [a - b(y_1 + y_2)] - c = 0 \Rightarrow y_2 = \frac{a-c}{2b} - \frac{1}{2}y_1$$

9

In Nash Equilibrium, each firm choose the optimal quantity produced given the production choice of the other firm.

Combine the two FOCs in part 8 we can get the Nash Equilibrium quantity profile:

$$y_1^* = y_2^* = \frac{a-c}{3b}$$

The industry production is $Y_D^* = y_1^* + y_2^* = \frac{2(a-c)}{3b}$.
The price level is $p_D^* = a - b(y_1^* + y_2^*) = a - \frac{2(a-c)}{3} = \frac{a+2c}{3}$.
The profit level of each firm is:

$$\pi_{1,D}^* = p_D^* y_1^* - c y_1^* = \frac{a+2c}{3} \cdot \frac{a-c}{3b} - c \cdot \frac{a-c}{3b} = \frac{(a-c)^2}{9b}$$

$$\pi_{2,D}^* = p_D^* y_2^* - c y_2^* = \frac{a+2c}{3} \cdot \frac{a-c}{3b} - c \cdot \frac{a-c}{3b} = \frac{(a-c)^2}{9b}$$

Then, the aggregate profit level is:

$$\Pi_D^* = \pi_{1,D}^* + \pi_{2,D}^* = \frac{2(a-c)^2}{9b}$$

10

The table below compares the equilibrium results among three market structures.

	Perfect Competition	Monopoly	Duopoly
Y^*	$\frac{a-c}{b}$	$\frac{a-c}{2b}$	$\frac{2(a-c)}{3b}$
p^*	c	$\frac{a+c}{2}$	$\frac{a+2c}{3}$
Π^*	0	$\frac{(a-c)^2}{4b}$	$\frac{2(a-c)^2}{9b}$

The conclusions are:

- $Y_{PC}^* > Y_D^* > Y_M^*$
- $\Pi_M^* > \Pi_D^* > \Pi_{PC}^*$
- $p_M^* > p_D^* > p_{PC}^*$

11

If there are I firms, all identical, with production costs $c(y) = cy$.

The profit maximization problem can be written as:

$$\max_{y_i} \quad \pi_i = p(Y)y_i - cy_i = (a - bY)y_i - cy_i = [a - b(y_i + y_{-i})]y_i - cy_i$$

where $y_{-i} = \sum_{j \neq i} y_j$.

The first order condition of a firm can be written as:

$$-by_i + [a - b(y_i + y_{-i})] - c = 0$$

12

Here, we look for a symmetric solution, which means $y^*_{-i} = \sum_{j \neq i} y^*_j = (I-1)y^*_O$.

Plug this into the FOC, we have:

$$-by^*_O + [a - b(y^*_O + (I-1)y^*_O)] - c = 0 \Rightarrow y^*_O = \frac{a-c}{(I+1)b}$$

Therefore, the Nash equilibrium quantity $y^*_O = \frac{a-c}{(I+1)b}$.

The industry production is $Y^*_O = Iy^*_O = \frac{I(a-c)}{(I+1)b}$.

The price level is $p(Y)^*_O = a - bY^*_O = \frac{a+Ic}{I+1}$.

The profit level of each firm is $\pi^*_O = p(Y)y^*_O - cy^*_O = \frac{(a-c)^2}{b(I+1)^2}$.

And the aggregate profit level is $\Pi^*_O = I\pi^*_O = \frac{I(a-c)^2}{(I+1)^2 b}$.

13

From the part 12, we can find that this general oligopoly solution embeds the previous cases:

When $I = 1$, we go back to monopoly, $p^*_M = \frac{a+c}{2}$, $\Pi^*_M = \frac{(a-c)^2}{4b}$, $Y^*_M = \frac{a-c}{2b}$.

When $I = 2$, we get the duopoly solution, $p^*_D = \frac{a+2c}{3}$, $\Pi^*_D = \frac{2(a-c)^2}{9b}$, $Y^*_D = \frac{2(a-c)}{3b}$.

When $I \to \infty$, the prices the total quantity produced, and the total industry profits converge to the perfect competition ones.

In detail,

$$\lim_{I \to \infty} p^*_O = \lim_{I \to \infty} \frac{a + Ic}{I+1} = c$$

$$\lim_{I \to \infty} Y^*_O = \lim_{I \to \infty} \frac{I(a-c)}{(I+1)b} = \frac{a-c}{b}$$

$$\lim_{I \to \infty} \Pi^*_O = \lim_{I \to \infty} \frac{I(a-c)^2}{(I+1)^2 b} = 0$$

In contrast to quantity competition, Bertrand competition focuses on price competition. When two firms compete in this setting, they will engage in a race to lower their prices, ultimately driving the equilibrium price down to marginal cost. As a result, both firms earn zero profit in equilibrium.

This outcome arises because if one firm earns a positive profit, the other firm can slightly undercut its price to capture the entire market share. This process continues until the price equals marginal cost ($p = c$), yielding the same outcome as perfect competition.

Therefore, we only need two firms to get the same outcomes as perfect competition.